1 Predicate Calculus

Predicate calculus contains all the components of propositional calculus, including propositional variables and constants. In addition, predicate calculus contains terms, predicates and quantifiers.

Definition 1 *Terms in predicate calculus have the same role as nouns in English.*

Definition 2 *Predicates are used to describe certain properties or relationships between individuals or objects.*

Definition 3 *Quantifiers indicate how frequently a certain statement is true. Specifically the **Universal Quantifier** indicates if a statement is **always** true usually using the for all symbol $\forall$. The **existential quantifier** indicates that a statement is sometimes true using the there exists symbol $\exists$.*

Definition 4 *The universe of discourse or **domain** is the collection of all persons, ideas, symbols, datastructures and so on, that affect the logical argument under consideration. The elements of the universe of discourse are called individuals. (p60) However in research these are often just called symbols or objects. To refer to objects we used identifiers called individual constants.*

In predicate calculus each predicate is given a name which is followed by a list of arguments. So the statement Mary and Paul are siblings would be written

$$siblings\,(mary, paul)$$

Many logicians use only single letter variables and would write this as follows.

$$s\,(m, p)$$

Not that this has an arity of 2. The statement Tom is a cat is written

$$cat\,(Tom)$$

has an arity of 1. A Predicate with an arity of one is called a **property**.

Definition 5 *A predicate name, followed by an argument list in parentheses, is called an **atomic formula**.*

Atomic formulas can be combined by logical connectives like propositions:

$$cat\,(Tom) \Rightarrow hastail\,(Tom) \tag{1}$$

We note that if all arguments of a predicate are individual **constants**, then *the resulting atomic formula must be true or false.*

Definition 6 *Any method that assigns truth values to all possible combinations of individulals of a predicate is called an **assignment** of the predicate.*

Example 7 *Suppose want to write an assignment table for four individuals (D, J, M, P) where the predicate mother has an arity of 2. Table 1 is an **assignment** for the predicate mother. By convention rows contain the first argument and columns contain the second argument.*

	D	J	M	P
D	F	F	F	F
J	F	F	T	T
M	F	F	F	F
P	F	F	F	F

Table 1

1.1 Variables and Instantiations

To allow more than just constants in predicate arguments we allow variables. Normally these are lower case alphabet characters. Using this convention we can restate statement (1) as follows.

$$cat(x) \Rightarrow hastail(x)$$

As in propositional calculus expressions can be given names.

$$A = cat(x) \Rightarrow hastail(x) \tag{2}$$

This means that when we write A we really mean $cat(x) \Rightarrow hastail(x)$. If the expression represented by A **contains** x, we say that A contains x.

Definition 8 *Let A represent an expression, x represent a variable, and t represent a term. Then $S_t^x A$ represents the expression obtained by replacing all occurences of x in A by t. $S_t^x A$ is called an **instantiation** of A, and t is said to be an **instance** of x.*

Example 9 *Therefore (1) could be written as $S_{Tom}^x A$ where A is given by (2).*

1.2 Quantifiers

Definition 10 *Let A represent an expression, and let x represent a variable. If we want to indicate that A is true for all possible values of x, we write $\forall xA$. Here, $\forall x$ is called the **universal quantifier**, and A is called the scope of the quantifier. The variable x is said to be **bound** by the quantifier. The symbol $\forall$ is pronounced "for all" or "for every."*

Example 11 *Suppose we want to write the statment "Everyone gets a break once in a while" using predicate calculus. We define B to be the predicate "gets a break once in a while." The word "everyone" indicates that this is true for all x. Thus we write the statements as $\forall xB(x)$.*

Example 12 *Consider the statement that all cats have tails. We can write this using predicate calculus as $\forall x (cat(x) \Rightarrow hastail(x))$.*

Definition 13 *Let A represent an expression, and let x represent a variable. If we want to indicate that A is true for at least one value of x, we write $\exists xA$. This statement is pronounced, "There exists x such that A." Here, $\exists x$ is called the **existential quantifier**, and A is called the **scope** of the existential quantifier. The variable x is said to be **bound** by the quantifier.*

Note: Both $\exists x$ and $\forall x$ should be treated like unary connectives.

Example 14 *Given a statment, "Some objects are blue." We can write this in predicate calculus as $\exists xBlue(x)$.*

Definition 15 *An expression is called a **variant** of $\forall xA$ if it is of the form $\forall yS_y^x A$, where y is any variable name, and $S_y^x A$ is the expression obtained from A by replacing all instances of x by y. Similarly, $\exists xA$ and $\exists yS_y^x A$ are variants of one another.*

Example 16 *Translate "Everybody has sombody who is his or her mother" into predicate calculus.*

Solution 17 *We define M to be the predicate "mother"; that is, $M(x, y)$ stands for x is the mother of y. The statement becomes*

$$\forall y \exists xM(x, y)$$

Example 18 *Translate "Nobody is perfect" into predicate calculus.*

Solution 19 *Let P be the property perfect, then we write $\neg\exists xP(x)$ or $\forall x\neg P(x)$*

In each definition we have stated that a variable is **bound**. That is $\forall xA(x)$, x is bound. But if you break this up and are only talking about $A(x)$, x would be free.

Definition 20 *If a variable is not bound, it is said to be free.*

Example 21 *Given* $\forall z\left(P\left(z\right)\wedge Q\left(x\right)\right)\vee\exists yQ\left(y\right)$ *find the bound and free variables. In this case* x *is free, and both* z *and* y *are bound in **all** occurances.*

Some times it is necessary to restrict the domain.

Example 22 *Consider the statement "All dogs are mammals." We can write this as If* x *is a dog, then* x *is a mammal. In predicate calculus we write* $\forall x\left(D\left(x\right)\Rightarrow M\left(x\right)\right)$.

Example 23 *Consider the statement "Some dogs are brown." We write this as* $\exists x\left(D\left(x\right)\wedge B\left(x\right)\right)$.

Example 24 *Consder the statement "only dogs bark." We write this as* $\forall x\left(bark\left(x\right)\Rightarrow dog\left(x\right)\right)$.

2 Interpretations and Validity

Valid expressions play the same role in predicate calculus that tautologies played in propositional logic. Generally speaking an expression A is *valid* if it is true for all *interpretations*.

2.1 Interpretations

Formally, an interpretation of a logical expression contains the following components:

1. There must be a universe of discourse (domain)
2. For each individual, there must be an individual constant that exclusively refers to this particular individual, and to no other.
3. Every free variable must be assigned a unique individual constant.
4. There must be an assignment for each predicate used in the expression, including predicates of arity 0, which represent propositions.

2.2 Validity

Definition 25 *An expression is **valid** if it is true under all interpretations. To express that an expression* A *is valid, we write* $\models A$.

All tautologies are valid expressions. The only difference between valid and tautology is that tautologies do not involve quantifiers or predicates. whereas valid expressions are not restricted in this way.

Definition 26 *If* B *is an expression, then any interpretation that makes* B *yield* T *is said to satisfy* B*. Any interpretation that satisfies* B *is called a **model** of* B*. If* B *has a model, then* B *is said to be **satisfiable**. Hence, an expression* A *is not valid if* $\neg A$ *is satisfiable. Equivalently, if* $\neg A$ *has a model then* A *cannot be valid.*

Definition 27 *An expression* B *that has no model is said to be contradictory.*

Definition 28 *Let* A *and* B *represent two expressions. We say that* A *is **logically equivalent** to* B *if* $A\iff B$ *is valid. In this case, we write* $A\equiv B$*. Moreover, we say* A *logically implies* B*, or* $A\Rightarrow B$ *is valid.*

Example 29 *Show the following.*

$$\forall x\left(P\Rightarrow Q\left(x\right)\right)\equiv P\Rightarrow\forall xQ\left(x\right) \tag{3}$$

Solution 30 *Here P is a propositional and Q is a predicate. We can use the law of cases to break this up.*

$$\forall x\left(T\Rightarrow Q\left(x\right)\right)\equiv T\Rightarrow\forall xQ\left(x\right). \tag{4}$$

$$\forall x\left(F\Rightarrow Q\left(x\right)\right)\equiv F\Rightarrow\forall xQ\left(x\right). \tag{5}$$

Since $T\Rightarrow Q\left(x\right)$ is $Q\left(x\right)$ and $T\Rightarrow\forall xQ\left(x\right)$ is $\forall xQ\left(x\right)$, Equation 4 becomes

$$\forall xQ\left(x\right)\equiv\forall xQ\left(x\right)$$

and since both $F\Rightarrow Q\left(x\right)$ and $F\Rightarrow\forall xQ\left(x\right)$ are both trivially true, we conclude that it doesn't matter if P is true or false.

2.3 Converting Valid expressions to schemas

Just like in propositional logic where you can convert tautologies to schemas, you can convert **valid** expressions to **schemas** except that you must give special attention to bound and free variables. You must take care not to substitute an expression that contains bound variables into the schema for one that does not contain bound variables. Consider equation (3) and let $H\left(x\right)$ be "x is happy," P stand for "the sun is shining" and Q for "the weather is nice." Then we can substitute $P\wedge Q$ for P and $H\left(x\right)$ for $Q\left(x\right)$ in (3). Thus the statement

$$\forall x\left(P\wedge Q\Rightarrow H\left(x\right)\right)\equiv\left(P\wedge Q\right)\Rightarrow\forall xH\left(x\right)$$

is correct. To see an example that is not correct let $S\left(x\right)$ be the statement that x sings. Then the statement

$$\forall x\left(S\left(x\right)\Rightarrow H\left(x\right)\right)\equiv\left(S\left(x\right)\right)\Rightarrow\forall xH\left(x\right) \tag{6}$$

clearly is not true. The left side says if a person sings, that person is happy, while the right side says that if any person sings, every one is happy. In actuality since x in $S\left(x\right)$ is a free variable equation (6) isn't really equation (3) at all.

2.4 Invalid Expressions

It only takes one, because an expression A is valid if and only if no **interpretation** yields F. One pitfall is that one would normally assume if $\forall xA$, then one could assume for $x=y$. That is $\forall xA\Rightarrow S_{y}^{x}AM$. However if y is bound in A, this is not true. Consider the following.

$$\forall x\exists yP\left(x,y\right)\Rightarrow\exists yP\left(y,y\right)$$

2.5 Proving Validity

The problem of proving whether or not an expresion is valid is undecidable!

Definition 31 *An undecidable problem has no general solution in the sense that there is no method that can reliably provide an answer to the problem.*

Some techniques include

1. - Show that the negation is contradictory
   - Show that the expression is a tautology

3 Derivations

3.1 Universal (Instantiation and Generalization)

Here we Introduce and define the rules to *insert and remove universal and existential quantifers* as well as a new concept called *unification.*

Definition 32 (Universal Instantiation - UI) *From* $\forall x P(x)$ *we can derive* $P(t)$ *for any term* t *as long as* t *is not bound in* P.

Example 33 $\forall x(cat(x) \Rightarrow hastail(x))$ *logically implies that* $cat(tom) \Rightarrow hastail(tom)$ *because tom is not a bound variable.*

Definition 34 (Universal Generalization - UG) *If* A *is any expression and if* x *is a variable that does not appear free in any premise, one has* $A \Rightarrow \forall x A$. *Here we say that the universal generalization is over* x. *Also* x *may not appear in any premise or it must be bound in all premises.*

Example 35 *Let* $P(x) =$ *"x is a computer scence major" and* $Q(x) =$ *"x likes programming" prove:* $\forall x P(x), \forall x (P(x) \Rightarrow Q(x)) \vdash \forall x Q(x)$

Solution 36

	Formal Derivation	Rule	Comment
1.	$\forall x P(x)$	Premise	Everyone is a CS major
2.	$\forall x (P(x) \Rightarrow Q(x))$	Premise	CS majors like programming
3.	$P(x)$	$1, S_x^x$	x is a CS major (instantiation)
4.	$P(x) \Rightarrow Q(x)$	$2, S_x^x$	If x is a CS major he likes programming
5.	$Q(x)$	$3, 4, MP$	x like programming
6.	$\forall x Q(x)$	UG	Everyone like programming

Example 37 (Using the deduction theorem with UG) *Let* $S(x)$ *stad for "x studied" and* $P(x)$ *stand for "x passed." The premise is that every one whostudied passed. Prove that everyone who did not pass, did not study.*

Solution 38

	Formal Derivation	Rule	Comment
1.	$\forall x (S(x) \Rightarrow P(x))$	Premise	Everyone who studied passed
2.	$S(x) \Rightarrow P(x)$	S_x^x	UI
3.	$\quad \neg P(x)$	Assumption	Assume that x did not pass
4.	$\quad \neg S(x)$	$2, 3, MP$	x cannot have studied
5.	$\neg P(x) \Rightarrow \neg S(x)$	DT	Apply DT and discharge $\neg P(x)$
6.	$\forall x (\neg P(x) \Rightarrow \neg S(x))$	$5, UG$	*

Note: Anyone who did not pass cannot have studied. This generalization is possible becasue x is not free in any premise.

Definition 39 *All variables that are not fixed will be called* ***true variables****. A variable may be universally generalize if and only if it is a true variable. If a variable appears in a premise, then it is assumed to be fixed, unless it is explicitly stated that the variable is true. We also note that true variables are strictly local to the line we use them on. This allows us to reuse* x *over and over again on different lines as different variables. Think scope.*

The result of definition 39 is that we can despense with the use of the universal quantifier in favor of using true variables *which in turn simplifies proofs.*

Notation 40 $x := y$ *is used to denote instantiation where* x *is replaced by* y *instead of* S_y^x. *So instead of writing "instantiate* x *to* a *in line* n*," we write "*n *with* $x := a$*."*

3.2 Unification

Definition 41 *Two expressions are said to unify if there are legal instantiations that make the expressions in question identical. The act of unifying is called* ***unification****. The instantiation that unifies the expressions in question is called a* ***unifier****.*

Example 42 $Q(a, y, z)$ *and* $Q(y, b, c)$ *are expressions appearing on different lines. Show that the two expression unify, and give a unifier. Here* a, b *and* c *are fixed, and* y *and* z *are true variables.*

Solution 43 *Since* y *in* $Q(y, b, c)$ *is a different variable than* y *in* $Q(y, b, c)$*, rename* y *in the second expression to become* y_1*. This means that one must unify* $Q(a, y, z)$ *with* $Q(y_1, b, c)$*. An instance of* $Q(a, y, z)$ *is* $Q(a, b, c)$*, and an instance of* $Q(y_1, b, c)$ *is* $Q(a, b, c)$*. Since these two instances are identical,* $Q(a, y, z)$ *and* $Q(y, b, c)$ *unify. The unifier is* $a = y_1, b = y, c = z$*.*

We want to make sure that we use the least number of unifiers possible. If two different solutions exist (and two or more ususally do) the one with the fewest uniferies is said to be more general.

3.3 Existintial (Instantiation and Generalization)

Definition 44 (Existential Generalization - EG) $S_t^x A \Rightarrow \exists x A$*. This just means that if I can find a term* t*, such that* $A(t)$ *is true, then three exists an* x *such that* $A(x)$*. If you are saying what? I don't get it, thats because it's too easy. If I can find one then there exists one.*

Consider the following informal example:

1. Everybody who has won a million is rich
2. Mary has won a million.
3. There is somebody who is rich.

or

1. $\forall x W(x) \Rightarrow R(x)$
2. $W(Mary) \Rightarrow R(Mary)$
3. $W(Mary)$
4. $R(Mary)$
5. $\exists x R(x)$

Let's try a more difficult derivation

Theorem 45 $\neg \exists x P(x) \vdash \forall x \neg P(x)$

Proof. ■

	Formal Derivation	Rule	Comment
1.	$\neg \exists x P(x)$	Premise	...
2.	$P(x)$	Assumption	Assume $P(x)$
3.	$\exists x P(x)$	2,EG	...
4.	$P(x) \Rightarrow \exists x P(x)$	DT	Discharge $P(x)$ and write 4
5.	$\neg P(x)$	1,4,MT	...
6.	$\forall x \neg P(x)$	3,UG	Since x is not a free variable in any premise

Definition 46 (Existential Instantiation - EI) $\exists x A \Rightarrow S_b^x A$*. This just says that if you have existential quantification, you can pick an instance but you must use a variable that has not appeared earlier as a free variable. The safest choice is to pick a new variable.*

Consider the following informal example:

1. Someone has won a million dollars
2. Everybody who has won a million is rich
3. There is somebody who is rich.

Theorem 47 $\forall x\left(W\left(x\right)\Rightarrow R\left(x\right)\right),\exists xW\left(x\right)\vdash\exists xR\left(x\right)$

Proof. ■

	Formal Derivation	Rule	Comment
1.	$\exists xW\left(x\right)$	Premise	Somebody won...
2.	$W\left(b\right)$	EI	Call the winner b
3.	$\forall x\left(W\left(x\right)\Rightarrow R\left(x\right)\right)$	Premise	
4.	$W\left(b\right)\Rightarrow R\left(b\right)$	S_b^x	If $W\left(x\right)\Rightarrow R\left(x\right)$ for everybody it holds for b
5.	$R\left(b\right)$	$2,4,MP$	
6.	$\exists xR\left(x\right)$	EG	Sombody is rich

4 Logical Equivalences

Table 4: Equivalences Involving Quantifiers

1.	$\forall xA\equiv A$	if x is not free in A
1d.	$\exists xA\equiv A$	if x is not free in A
2.	$\forall xA\equiv\forall yS_y^xA$	if y is not free in A
2d.	$\exists xA\equiv\exists yS_y^xA$	if y is not free in A
3.	$\forall xA\equiv S_t^xA\wedge\forall xA$	for any term t
3d.	$\exists xA\equiv S_t^xA\vee\exists xA$	for any term t
4.	$\forall x\left(A\wedge B\right)\equiv A\vee\forall xB$	if x not free in A
4d.	$\exists x\left(A\vee B\right)\equiv A\wedge\exists xB$	if x not free in A
5.	$\forall x\left(A\wedge B\right)\equiv\forall xA\wedge\forall xB$	
5d.	$\exists x\left(A\vee B\right)\equiv\exists xA\vee\exists xB$	
6.	$\forall x\forall yA\equiv\forall y\forall xA$	
6d.	$\exists x\exists yA\equiv\exists y\exists xA$	
7.	$\neg\exists xA\equiv\forall x\neg A$	
7d.	$\neg\forall xA\equiv\exists x\neg A$	

Definition 48 *Renaming the variables in an expression such that distict variables have distinct names is called* ***standardizing the variables apart****.*

There are lots of examples where we can use the rules in table 4. If you would like to see some of them see section 2.4 in [[1]]

5 Equational Logic

Axiom 49 (Reflexivity) $\forall x\left(x=x\right)$

Notation 50 *If A is any expression, the $R\left(n\right)_r^tA$ is the expression one otains from A by replacing the n^{th} instance of term t by r. If t occurs fewer than n times in A, then $R\left(n\right)_r^tA=A$.*

Axiom 51 (Substitution Rule) *If A and $t=r$ are two expressions that have been derived, one is allowed to conclude that $R\left(n\right)_r^tA$ for any $n>0$. In this case we say that we substitute t from $t=r$ into A.*

One thing of note is that the substitution rule can be applied to subexpressions embedded in other expressions. No other rule of inference discussed so far can do this. Thus the substitution rule is very effecient.

5.1 Equality and Uniqueness

Consider the following:

$$\begin{aligned} lion &= mammal \\ bear &= mammal \end{aligned}$$

Certainly both the lion and the bear are mammals, but this is incorrect because we could conclude

$$lion = bear$$

which is obviously incorrect. Thus we can not assume that the word "is" can always be translated as equals. This makes sense when you think of their definition of the predicate $=$ as a function. This naturally presupposes that there can be only one x for each y such that $x = y$.

Notation 52 $\exists_1 x P(x)$ *indicates that there is exacltly one* x *that makes* $P(x)$ *true.*

References

[1] Logic and Discrete Mathematics A computer Science Perspective by Winfried Karl Grassmann, Jean-Paul Tremblay