Cauchy-Schwarz $|v \cdot v| \leq ||u|| + ||v||$. Triangle Inequality $||u+v|| \leq ||u|| + ||v||$. Distance $d(u,v) = ||u-v||$. $\cos\theta = \frac{u \cdot v}{||u|| ||v||}$. $||u+v||^2 = ||u||^2 + ||v||^2$ iff u is orthoganol to v. $proj_u V = (\frac{v \cdot u}{u \cdot u})u = \frac{v \cdot u}{||u||^2} u$. Where p is a point on the line, n_i is a normal vector to the line, and d is a direction vector on the line: Normal Form: line→ $\begin{cases} n_1 \cdot x = n_1 \cdot p_1 \\ n_2 \cdot x = n_2 \cdot p_2 \end{cases}$ plane→ $n \cdot x = n \cdot p$. General Form: line→ $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$. plane→ $ax + bx + cz = d$. Vector Form: line→ $x = p + td$. plane → $x = p + su + tv$. Parametric Form (Just expand the Vector form into three equations. Homogeneous system → $Ax = 0$. Distance from point $B = (x_0, y_0, z_0)$ to plain $ax + by + cz = d$ is $\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$. Unique infinite solution(s) → consistent, No solutions →inconsistent. A matrix is in row echelon form if it satisfies the following properties: Any row consisting entirely of zeros is at the bottom. In each nonzero row, the first nonzero entry is in a column to the left of any leading entries below it. The **rank** of a matrix is the number of nonzero rows in its row echelon form. Rank Theorem # of free variables $v = n - Rank(A)$ where n is the number of variables. reduced row echelon form: 1)It is in row echelon form. 2) The leading entry in each nonzero row is a 1. 3) Each column containing a leading 1 has zeros everywhere else. 2.6: Let $v_1, v_2, \ldots, v_m$ be column vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[v_1 \ldots v_m]$ with these vectors as its cols. Then $v_1, \ldots, v_m$ are lin. dep. if and only if the homogeneous linear system $[A|0]$ has a nontrivial solution. 2.7: Given $[v_1 \ldots v_m]^T$ then $v_1, \ldots, v_m$ are linearly depedent if and only if rank$(A) < m$. Symmetric → $A = A^T$. Algebraic Properties of Matricies include: (+,scalary) Commutativity, Asscociativity, Identity, +inverse, distributivity Transpose Properties Theorem 3.4: $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(kA)^T = k(A^T)$, $(AB)^T = B^T A^T$, $(A^{+r})^T = (A^T)^{+r}$ Theorem 3.5: If A is a square matrix, then $A + A^T$ is a symmetric matrix. For any matrix A, AA^T and $A^T A$ are symmetric matrices. T 3.6: Inverses of square matrices are unique. T 3.7: A invertable → $Ax = b$ has a unique solution. T 3.9: A invertable → A^{-1} invertable. $(cA)^{-1} = \frac{1}{c} A^{-1}$. $(AB)^{-1} = B^{-1} A^{-1}$. $(A^T)^{-1} = (A^{-1})^T$. $(A^n)^{-1} = (A^{-1})^n$. $A^{-n} = (A^{-1})^n$. $(A_1 \times \ldots x A_n)^{-1} = A_n^{-1} \times \ldots \times A_1^{-1}$. T 3.10: Let E be an elementary matrix, then performing that operation on A is the same as EA. T 3.11: Each elementary matrix is invertible and its inverse is an elementary matrix of the same type. Def: Let A be a square matrix. A factorization of $A = LU$, where L is **unit** lower triangular and U is upper triangular, is called an LU factorization. Consider $Ax = b$, and then $Ax = LUx = L(Ux) = b$. Let $LU = y$. Now solve $Ly = b$, and then $Ux = y$. Find LU factorization: Row reduce to ref form using elementary (in order top to bottom column by column) row ops of the form $R_{\text{row}} - kR_{\text{col}}$. Then place k in the identity matrix at row, col. T 3.16: If A is an invertable matrix that has an LU factorization, then L and U are unique. Def: A subspace of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that (1) $0 \in S$, (2) closed under addition and (3) scalar multiplication. T 3.19: For $v_{i=1,k} \in \mathbb{R}^n$, $span(v_1, \ldots, v_k)$ is a subspace of $\mathbb{R}^n$. Def: row space of A is the subspace $\text{row}(A)$ spanned by the rows of A. Def: col space of A is the subspace $\text{col}(A)$ spanned by the rows of A. T 3.20: Let B be any matrix that is row equivalent to a matrix A. Then $\text{row}(A) = \text{row}(B)$. (Two matrices are row equivalent iff they can be reduced to the same row eschlon form.) T 3.21: Let A be an $m \times n$ matrix and let N be the set of solutions to $Ax = 0$. Then N is a subspace called the null space. Def: A basis for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that (1) spans S and (2) is Lin. Ind.. To find the $\text{row}(A), \text{col}(A), null(A)$: (1) find $R = rref(A)$. (2) Use the nonzero row vectors of R containing leading 1s to form a basis for $\text{row}(A)$. (3) Use the column vectors of A with leading $1s$ in R as a basis for $\text{col}(A)$. (4) use $Rx = 0$ to find the $null(A)$. T 3.23: Any two bases for S have the same number of vectors. Def: $\dim(S)$ is the number of vectors in the basis for S. $\dim(\text{row}(A)) = \dim(\text{col}(A)) = rank(A) = rank(A^T)$. $nullity(A) = \dim(null(A))$. Let A be an $m \times n$ matrix, then $rank(A) + nullity(A) = n$. T 3.28: $A : m \times n$. (1) $rank(A^T A) = rank(A)$. (2) $A^T A$ is invertable iff $rank(A) = n$. Def: Let S be a subspace of $\mathbb{R}^n$ and let $\mathcal{B}$=$\{v_1, \ldots, v_k\}$ be a basis for S. Let $v \in S$, $v = c_1 v_1 + \ldots + c_k v_k$ where $v_{\mathcal{B}} = \begin{bmatrix} c_1 & \ldots & c_k \end{bmatrix}^T$ is called the coordinate vector of v with respect to $\mathcal{B}$. Def: $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear tranformation if $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$. $[T] = [T(e_1)| \ldots |T[e_n]]$. $(S \circ T)(v) = S(T(v)) = [S][T]v$. Two transformations are considered inverses if $S \circ T = T \circ S = I_n$. For these we have $[T^{-1}] = [T]^{-1}$. Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector x such that $Ax = \lambda x$. Such vectors e_λ are called eigenvectors. The collection of all eigenvectors associated with a λ is called the eigenspace denoted $E_\lambda = span(\{e_\lambda\})$ or $E_\lambda = \{t \cdot e_\lambda\}$. T 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal. T 4.3: Let A be an $n \times n$ matrix. (1) If A has a zero row (column) then $\det(A) = 0$. (2) If B is obtained by interchanging two rows (or cols) then $\det(B) = -\det(A)$. (3) If A has two identical rows (or cols) $\det(A) = 0$. (4) If B is obtained by multiplying a row of A by k then $\det(B) = k\det(A)$. (5) If A, B, C are identical except that the i^{th} row (column) of C is the sum of the i^{th} rows (cols) of A and B, then $\det(C) = \det(A) + \det(B)$. If B is obtained by adding a multiple of one row (column) of A to another row (column) then $\det(B) = \det(A)$. T 4.8: $A, B : n \times n$. Then $\det(AB) = \det(A)\det(B)$. T 4.7: $A : n \times n$. $\det(kA) = k^n \det(A)$. T 4.9: If A is invertable, $\det(A^{-1}) = \frac{1}{\det(A)}$. T 4.10: $\det(A) = \det(A^T)$. T 4.11: $A : n \times n$ invertable and $Ax = b \to x_i = \frac{\det(A_i(b))}{\det(A)}$ where $A_i(b)$ means replace the i^{th} column by b. Def: The adjoint is the transpose of the matrix of cofactors (the $\det(submatrix)$) So $C_{11} = \det\begin{bmatrix} a_{22} \ldots a_{2n} ; a_{n2} \ldots a_{nn} \end{bmatrix}$. $\det(A - \lambda I) = 0$ (Characteristic Polynomial). To find the eigenvectors find the null space of $A - \lambda I$ also called the eigenspace E_λ. Find the basis for the space. Define **algebraic multiplicity** to be its multiplicity as a root of the characteristic equation. Define **geometric multiplicity** of an eigenvalue to be $\dim(E_\lambda)$. T 4.15 The eigenvalues of a triangular matrix are the entries on its main diagonal. T 4.16 A square matrix A is invertable iff 0 is *not* an eighenvalue of A. T 4.17: Let A be an $n \times n$ matrix then: A is invertable $\leftrightarrow$ $Ax = b$ has a unique solution $\leftrightarrow$ $Ax = 0$ only has a trivial solution $\leftrightarrow$ $rref(A) = I \leftrightarrow A$ is the product of elementary matrices $\leftrightarrow$ $rank(A) = n \leftrightarrow nullity(A) = 0 \leftrightarrow$ the column vectors of A are Lin. Ind. $\leftrightarrow$ the column vectors span $\mathbb{R}^n$ $\leftrightarrow$ the column vectors are a basis for $\mathbb{R}^n$ $\leftrightarrow$ rows(A) are Lin. Ind. $\leftrightarrow$ rows(A) span $\mathbb{R}^n$ $\leftrightarrow$ rows(A) form a basis for $\mathbb{R}^n$ $\leftrightarrow \det(A) \neq 0 \leftrightarrow 0$ is not an eigenvalue of A. T 4.18: Given $A \in M_{n \times n}$ with eigenvalue λ and eigenvector x, then (a) For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x (b) If A is invertable then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector x (c) For any integer n, λ^n is an eigenvalue of A^n with eigenvector= x. T4.19: $A \in M_{n \times n}$, with eigenvectors $v_1, \ldots v_m$. If $x \in R^n$ that can be expressed as $x = c_1 v_1 + \ldots + c_m v_m$, then for any integer k, $A^k x = c_1 \lambda_1^k v_1 + \ldots + c_m \lambda_m^k v_m$. **4.4**. D: A is **similar** to B if ther is an invertable matrix P such that $P^{-1}AP = B$ denoted $A \sim B$. Note P is not unique, let $A = I$. $\sim$ is an equivalence relation ($A \sim A, A \sim B \to B \sim A, A \sim B$ and $B \sim C \to A \sim C$). T4.22: If $A \sim B$ then (a) $\det(A) = \det(B)$, (b) A is invertable iff B is invertable, (c) $rank(A) = rank(B)$, (d) A and B have the same characteristic polynomial, (e) A and B have the same eigenvalues. D: An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that $A \sim D$ or $P^{-1}AP = D$. T4.23 A **is diagonalizable if and only if** A **has** n **Lin. Ind. eigenvectors**. **Where** P **is made up of the Lin. Ind. eigenvectors and the entries of** D **are made up of the corresponding eigenvalues**. T4.24: If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A and $\mathcal{B}_i$ is the basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$ is Lin. Ind.. T4.25: If $A_{n \times n}$ has n distict eigenvalues, then A is diagonalizable. L4.26: If $A_{n \times n}$, then for each eigenvalue, the geometric multiplicity $\leq$ algebraic multiplicity. T4.17: $A_{n \times n}$ is diagonalizable $\leftrightarrow$ $\mathcal{B}$ in (T4.24) contains n vectors. algebraic multiplicity = geometric multiplicity for each eigenvalue. **5.1**: D **Orthogonal set** if $v_i \cdot v_j = 0 \; \forall i \neq j$. T5.1: If $\{v_1, \ldots v_k\}$ is an orthogonal set, then the vectors are Lin. Ind.. D: An orthogonal basis for a subspace is also an orthogonal set. T5.2: Let $\{v_{i \ldots k}\}$ be an orthogonal basis for W. $w \in W \to w = c_1 v_1 + \ldots + c_k v_k$ where $c_i \; (w \cdot v_i)/(v_i \cdot v_i)$ (projection). D: An orthonormal set is an orthogonal set of unit vectors. An orthonormal basis for a subspace W of R^n is a basis of W that is an orthonormal set. T5.3: Let $\{q_{1 \ldots k}\}$ be an orthonormal basis for W. Then $w \in W \to w = (w \cdot q_1)q_1 + \ldots + (w \cdot q_k)q_k$ is unique. T5.4: **The cols of an** $m \times n$ **matrix** Q **form an orthonormal set iff** $Q^T Q = I_n$. D:An $n \times n$ matrix whose cols form an orthonormal set is called an **orthogonal matrix** → $Q^{-1} = Q^T$. T5.6: Q is orthogonal $\leftrightarrow ||Qx|| = ||x|| \leftrightarrow Qx \cdot Qy = x \cdot y$. T5.7: Q orthogonal matrix → rows form an orthonormal set. T5.8: Let Q be an orthogonal matrix then (a) Q^{-1} is orthogonal. (b) $\det(Q) = \pm 1$. (c) If λ is an eigenvalue of Q then $|\lambda| = 1$. (d) If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$. **D** Let W be a subspace of R^n. We say that a vector v in R^n is orthogonal to W if v is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the ortogonal complement of W, denoted $W^\perp$. That is $W^\perp = \left\{ v \in R^n : v \cdot w = 0 \text{ for all } w \in W \right\}$. T5.10: $(\text{row}(A))^\perp = null(A)$ and $(\text{col}(A))^\perp = null(A^T)$. D: $proj_W(v) = proj_{u_1}(v) + \ldots + proj_{u_k}(v)$. Then $perp_W(v) = v - proj_W(v)$. T5.11: $v = w + w^\perp$ where w and $w^\perp$ are unique. T5.13: If W is a subspace of R^n then $\dim W + \dim W^\perp = n \to rank(A) + nullity(A) = n$. T5.15: Gram-Schmidt process: Let $\{x_1, \ldots, x_n\}$ be a basis for a subspace W of R^n and define the following: $v_1 = x_1$, $v_k = x_k - \sum_{i=1}^{k-1} ((v_i \cdot x_i)/(v_i \cdot v_i))v_i$ for each $1 < i \leq n$. The the set $\{v_1, \ldots, v_n\}$ is an orthogonal basis for W. D: A square matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$. T5.17 If A is diagonalizable, then A is symmetric: $Q^T AQ = D \to A = QDQ^T$, $A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$. T5.18If A is a real symmetric matrix, then the eigenvalues of A are real. T5.19: If A is symmetric, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal. T5.20 (Spectral Theorem) Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable. (Spectral Decomposition): $A = QDQ^T = [q_1 \ldots q_n]\lambda_{1..n} IQ^T = \lambda_1 q_1 q_1^T + \ldots + \lambda_n q_n q_n^T$. **6.1**: A vector space is a field $(V, +, \cdot)$ (1)+closure, (2)+commutativity (3)+associativity (4)$\exists 0 \in V$ (5)additive inverses $u + (-u) = 0$ (6)•closure (7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (8)$(c+d)\vec{u} = c\vec{u} + d\vec{u}$ (9) $c(d\mathbf{u}) = (cd)\mathbf{u}$ (10) $1u = u$. D: Vector subspace has the same def= closure under addition and scalar multiplication. **6.2** D: linear dependence: Given $\{v_{1..k}\}$ then for $c_1 \ldots c_k$ at least one of which is non-zero $c_1 v_1 + \ldots + c_k v_k = 0$ →lin. dep.. D: A set S of vectors in a vector space V is lin. dep. if it contains finitely many lin. dep. vectors. T6.6: $[u+v]_B = [u]_B + [v]_B$, $[cu]_B = c[u]_B$. T6.7: Let $B = \{v_{1..n}\}$ be a basis for a vector space V and let $u_1, \ldots, u_k$ be vectors in V. Then $\{u_1, \ldots, u_k\}$ is Lin. Ind. in V if and only if $\{[u_1]_B, \ldots, [u_k]_B\}$ is Lin. Ind. in R^n. T6.10: Let V be a vector space with $\dim V = n$. Then (a) Any Lin. Ind. set in V contains at most n vectors. (b) Any spanning set for V contains at least n vectors. (c) Any Lin. Ind. set of exactly n vectors in V is a basis for V. (d) Any spanning set for V consisting of exactly n vectors is a basis for V. (e) Any linearly indpendent set in V can be extended to a basis for V. (f) Any spanning set for V can be reduced to a basis for V. 6.3: Let $\mathcal{B}$ =$\{u_{1..n}\}$, $\mathcal{C}$ =$\{v_{1..n}\}$ be bases for a vector space V. The $n \times n$ matrix whose cols are the coordinate vectors $[u_1]_C, \ldots, [u_n]_C$ of vectors in $\mathcal{B}$ with respect to C is denoted by $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and is called the change of basis matrix from B to C: $P_{C \leftarrow B} = [u_1]_C, \ldots, [u_n]_C$. (a) $P_{C \leftarrow B}[x]_B = [x]_C$ in V. (b) $P_{C \leftarrow B}$ is the unique matrix P with the property that $P[x]_B = [x]_C$ for all $x \in V$. (c) $P_{C \leftarrow B}$ is invertible and $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$. Gauss-Jordon Method: Let $C = P_{\mathcal{E} \leftarrow \mathcal{C}}$, and $B = P_{\mathcal{E} \leftarrow \mathcal{B}}$. Then $[C|B] \to [I|P_{\mathcal{C} \leftarrow \mathcal{B}}]$. D: Let $T : V \to W$ be a linear transformation. $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$, $range(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$, T6.18 $\ker(T)$ is a subspace, $range(T)$ is a subspace, D: $rank(T) = \dim(range(T))$, $nullity(T) = \dim(\ker(T))$, T6.19: $rank(T) + nullity(T) = \dim(V)$. D: A linear transformation $T : V \to W$ is called **1-to-1** if T maps distinct vectors in V to distinct vectors in W. If the $range(T) = W$, then T is called **onto**. T is 1-to-1 if $\forall u, v$ $u \neq v \Rightarrow T(u) \neq T(v)$ or $T(u) = T(v) \Rightarrow u = v$. T is **onto** if $\forall w \in W$, $\exists v \in V$ s.t. $w = T(v)$. T6.20: A linear transformation $T : V \to W$ is 1-to-1 iff $\ker(T) = \{\mathbf{0}\}$. T6.21: Let $\dim(V) = \dim(W) = n$. Then a lin. trans. $T : V \to W$ is 1-to-1 iff it is onto. Let $T : V \to W$ be a 1-to-1 linear transformation, If S is a set of Lin. Ind. vectors in V, then $T(S)$ is a set of Lin. Ind. vectors in W. (do it with $\dim V = \dim W = n$, and $|S| = n$, you get a basis to a basis.) T6.24: A Lin. Trans. is invertible iff it is 1-to-1 and onto. D: A Lin Trans. is called a **isomorphism** if it is 1-to-1 and onto. If $\exists T : V \to W$, 1-to-1, onto, then V is isomorphic to W denoted $V \cong W$. T6.25 V is isomorphic to W iff $\dim V = \dim W$.