1 Section 1.4

$\sqrt{-1}$ is not an integer, rational or real number. Yet we can express $x = \sqrt{-1}$ using algebra operations and real numbers. These numbers are used to explain advanced concepts in mathematics and physics. The field of study in Physics that uses imaginary numbers has produced technology that you use all the time. CD's, and other electronic gadgets.

Definition 1 *The imaginary unit is defined as* $i = \sqrt{-1}$.

Thus we have

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^2 \cdot i^2 = -1 \cdot -1 = 1 \\ i^5 &= i^4 \cdot i = 1 \cdot i = i \end{aligned}$$

Using this simple list of powers we can know immediately that

$$\begin{aligned} i^{35} &= \left(i^5\right)^7 = 1^7 = 1 \\ i^{128} &= i^{125} \cdot i^3 = 1 \cdot i^3 = -1 \end{aligned}$$

But what if we have something like $\sqrt{-36}$? We know that $\sqrt{36} = 6$, but what can we do about $\sqrt{-36}$?

$$\begin{aligned} \sqrt{-36} &= \sqrt{36 \cdot -1} \\ &= \sqrt{36} \cdot \sqrt{-1} \\ &= 6i \end{aligned}$$

Mixing reals and imaginary numbers using addition leads us to the definition of complex numbers.

Definition 2 *Complex numbers are the set of all numbers of the form*

$$a + bi$$

where $a, b \in \mathbb{R}$. *We denote the set of complex numbers as* $\mathbb{C}$. ***We call** a **the real part and** b **the imaginary part.***

Lemma 3 $\mathbb{R} \subset \mathbb{C}$.

Proof. Let $a \in \mathbb{R}$ (read this as: let a be an elment of the reals). Then we can write x as $x = a + 0i$. Clearly $x = a$ and $x \in \mathbb{C}$. Thus $\mathbb{R} \subset \mathbb{C}$. ■

Definition 4 *Given two complex numbers* $x = a + bi$, *and* $y = c + di$, $x = y$ *if and only if* $a = c$ *and* $b = d$.

Example 5 *Let*

$$\begin{aligned} x &= 1 + 2i \\ y &= 2 + 4i \end{aligned}$$

is $x = y$*? (no)*

Let

$$\begin{aligned} x &= 1 + 0i \\ y &= 1 \end{aligned}$$

is $x = y$*? (yes - just because I didn't write the* $0i$ *added to* y *doesn't mean they don't have the same meaning)*

Definition 6 *When adding or subtracting complex numbers, we add(or subtract) the real parts to find the new real part and we add (or subtract) the imaginary parts to find the new imaginary part.*

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

DO THE EXAMPLES FROM THE BOOK

Definition 7 *Complex multiplication follows the same pattern as polynomial multiplication where i is treated like a variable and then simplified using the power rules given above. The easiest way is to foil numbers as follows:*

$$\begin{aligned}(a+bi)\cdot(c+di) &= ac+ad\ i+bc\ i+bd\ i^2\\ &= ac+(ad+bc)i+bd(-1)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

DO THE EXAMPLES FROM THE BOOK

It is not possible to divide complex numbers directly. Instead we must find a way to remove the complex number from the denominator. To do this we define the conjugate of a complex number.

Definition 8 *Given a complex number $x=a+bi$ the complex conjugate of x is $a-bi$. Multiplying a number by its conjugate* ***always*** *results in a real number. (Hint if you don't get a real number you made a mistake).*

$$\begin{aligned}(a+bi)\cdot(a-bi) &= a^2-ab\ i+ab\ i-b^2\ i^2\\ &= a^2+(-ab+ab)i-b^2(-1)\\ &= a^2+b^2\end{aligned}$$

DO THE EXAMPLES FROM THE BOOK

Definition 9 *For any positive number b, the* ***principal square root*** *of the negative number $-b$ is defined by*

$$\sqrt{-b}=\sqrt{b}\ i$$

Example 10 *Recall that if $x^2=4$, $x=\pm 2$. (See example at the top of page 127).*