First-order Predicate Logic

INTRODUCTION

The language of propositional logic is obviously insufficient for more subtle reasoning. Here is an example:

"A prime number is divisible by no other number than 1 and itself, and the number 6 is divisible by 2. Hence, there is a number which is not prime."

All we can express (of this reasoning) in propositional logic is a coarse skeleton:

$$\varphi \wedge \psi \longrightarrow \sigma$$

In predicate logic we can express all the details:

$$(\forall\, x(Prime(x) \longrightarrow (\forall\, y\,(Divisible(x,\, y) \longrightarrow (y = 1 \vee y = x)))) \wedge Divisible(6,\, 2)) \;\longrightarrow\; (\exists\, z \,\neg\, Prime(z))$$

The power of predicate logic can be attributed to a supply of symbols for

1. **Relations and Properties**

   e.g. "$Divisible(x, y)$" and "$Prime(x)$"

2. **Quantifiers**

   "$\forall\, x\ \ldots$" "$\exists\, z\ \ldots$"

SYNTAX

Some typical formulas in predicate logic:

$\forall x \exists y (x = y^2 \longrightarrow y = 2x)$

$(\forall x (Prime(x) \longrightarrow (\forall y (Divisible(x, y) \longrightarrow (y = 1 \lor y = x)))) \land Divisible(6, 2)) \longrightarrow (\exists z \neg Prime(z))$

Such **formulas** are meant to express that a certain object or some unspecified ditto or maybe all objects have a certain property or that two or more objects are related to each other in some way.

The **formulas** are built with the help of symbols from a rich Alphabet. (See below.)

THE ALPHABET

LOGICAL symbols

Variables (countably many) $x, y, z, x_0, y_0, z_0, \ldots$ representing arbitrary individuals.

Connectives $\perp, \neg, \land, \lor, \longrightarrow, \longleftrightarrow$

Quantifiers $\forall, \exists$

Equality $=$

Auxiliary symbols (,), ,

NON-LOGICAL symbols

Constant symbols E.g. π, 0, 1, 2, Charles and c_i for an unspecified constant. These symbols are used to represent fixed individuals.

Predicate symbols are used to represent relations. E.g. we could write $x < y$ and $Prime(x)$ and $Divisible(x, y)$. In these cases $<$, *Prime* and *Divisible* are predicate symbols. A predicate symbol for an unspecified relation is often written P.

Function symbols which represent functions. E.g. we could write $x + y$ or $plus(x, y)$ and $mother(x)$. In these cases $+$, *plus*, *mother* are function symbols. A function symbol for unspecified function is often written f.

EXAMPLE 1 The **marked** symbols below are **logical**.

The unmarked are nonlogical.

$$\forall x \exists y (x = times(y, y) \longrightarrow y = plus(x, x))$$

$$(\forall x (Prime(x) \longrightarrow (\forall y (Divisible(x, y) \longrightarrow (y = 1 \lor y = x)))) \land Divisible(6, 2)) \longrightarrow (\exists z \neg Prime(z))$$

Different non-logical symbols are used in different reasonings.

THE TERMS (represent individuals)

1. Every **constant** is a term.

2. Every **variabel** is a term.

3. If f is a n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $\boldsymbol{f(t_1, \ldots, t_n)}$ is a (compound) term.

EXAMPLE 2 The marked strings are terms.

$$\forall\, \underline{x}\, \exists\, \underline{y}\big(\underline{x} = \underline{times(\underline{y}, \underline{y})} \longrightarrow \underline{y} = \underline{plus(\underline{x}, \underline{x})}\big)$$

$$(\forall\, \underline{x}(Prime(\underline{x}) \longrightarrow (\forall\, \underline{y}\,(Divisible(\underline{x}, \underline{y}) \longrightarrow (\underline{y} = \underline{1} \lor \underline{y} = \underline{x})))) \land Divisible(\underline{6}, \underline{2})) \longrightarrow (\exists\, \underline{z} \neg Prime(\underline{z}))$$

THE FORMULAS

1. $\bot$ is an atomic formula.

2. If t, u are terms, then $\boldsymbol{t = u}$ is an atomic formula.

3. If P is a n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $\boldsymbol{P(t_1, \ldots, t_n)}$ is an atomic formula. Also $\boldsymbol{P}$ alone is an atomic formula in case $\boldsymbol{P}$ is 0-ary.

4. If φ is a formula, then $\boldsymbol{\neg \varphi}$ is a formula.

5 If φ, ψ are formulas and $\Box \in \{\land, \lor, \longrightarrow, \longleftrightarrow\}$, then $\boldsymbol{(\varphi \Box \psi)}$ is a formula.

6 If φ is a formula and x is a variable, then $\boldsymbol{\forall x \varphi}$ and $\boldsymbol{\exists x \varphi}$ are formulas.

EXAMPLE 3 Formulas and binary tree-structure.

Every subtree corresponds to a subformula. In the leaves You find atomic formulas.